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Anomaly Problem in a Simple Model Analogous to the Lightcone-Gauge Two-Dimensional Quantum Gravity

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Abstract

It was found in the two-dimensional quantum gravity both in the de Donder gauge and in the lightcone gauge that one of field equations breaks down at the level of the representation, though the breakdown is very little. It is shown that this anomalous behavior occurs also in a very simple model analogous to the lightcone-gauge two-dimensional quantum gravity. This model, however, can be transformed into a free field theory by a nonsingular transformation. Of course, the latter has no anomaly. The cause of the discrepancy is analyzed.

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1. Introduction

Although the conventional covariant perturbation theory has been very useful in particle physics, it is *not adequate* to apply it to quantum gravity because to do so one must introduce a *wrong* assumption at the starting point [1]. That is, it is *not adequate* to solve quantum gravity in the interaction picture. We have therefore developed a new approach to the covariant quantum field theory for solving it *in the Heisenberg picture* [2–11]. The outline of our method is as follows.

First, from field equations and equal-time commutation relations, we set up Cauchy problems for all full-dimensional commutators. We then solve them (if necessary, we expand them in powers of parameters involved). From this operator algebra, we calculate all multiple commutators. Finally, we construct all n-point Wightman functions so as to be consistent with all multiple commutators under the requirement of energy positivity and the generalized normal-product rule. The latter is necessary to define Wightman functions involving composite operators (i.e., products of field operators at the same spacetime point).

Our method has been most successful in the two-dimensional quantum gravity in the de Donder gauge [3–7, 12–15]. We obtain a complete set of Wightman functions explicitly in the manifestly covariant and BRS-invariant way. Our solution is quite satisfactory, but it exhibits anomaly in a field equation, though all right if it is once covariantly differentiated.

Our method is applicable also to noncovariant theories. We have recently solved the two-dimensional quantum gravity in the lightcone gauge [16],[3] which is a local version of Polyakov's "induced" quantum gravity [17]. We have obtained the explicit solution, which satisfies Polyakov's Ward-Takahashi (WT) identities. Nevertheless, it exhibits anomaly: A field equation, which is a *second-order* differential equation, breaks down, while Polyakov's equation, which follows from it and is a *third-order* differential equation, does not suffer from anomaly.

In the present paper, we wish to analyze the problem of this new type of anomaly by using a very simple model. This model is obtained by simplifying the lightcone-gauge two-dimensional quantum gravity discussed in I, and it again exhibits anomaly of the same type. Interestingly enough, however, *this model is equivalent to a free-field theory,* that

[3] This paper is referred to as I hereafter.

is, there is a nonsingular transformation of field operators connecting both theories. Of course, a free-field theory is free of anomaly. Thus we encounter a kind of dilemma. We analyze its cause and find that the discrepancy is caused by the violation of the associative law for a product of distributions.

The present paper is organized as follows. In Section 2, we briefly review the two-dimensional quantum gravity in the lightcone gauge. In Section 3, we propose a simple model analogous to it and show that this model has anomaly similar to the one found in the two-dimensional quantum gravity. In Section 4, we point out that this model is equivalent to a free-field theory, which has no anomaly, and analyze why such a dilemma happens. In Section 5, we investigate the WT identities. The final section is devoted to discussion. In the Appendix, we discuss the symmetry properties of our model.

2. Review of the lightcone-gauge model

We first consider the two-dimensional quantum gravity in the lightcone gauge. Only one component of the gravitational field $g_{\mu\nu}$ survives; it is denoted by h ($= h_{++}$ in Polyakov's notation). Polyakov's Lagrangian density is a nonlocal one [17]; by introducing an auxiliary field $\tilde{b}$, it is rewritten into a local one. With the lightcone coordinates x^+ and x^-, the relevant Lagrangian density is written as

$$\mathcal{L} = \partial_- h \cdot \partial_- \tilde{b} - \frac{1}{2}\alpha h(\partial_- \tilde{b})^2 + \frac{1}{2}\gamma\alpha\partial_+ \tilde{b} \cdot \partial_- \tilde{b}, \tag{2.1}$$

where $\gamma = 1$ and α is an arbitrary nonzero real constant. For convenience, we generalize $\gamma = 1$ to γ arbitrary.

In I, we have completely solved the model defined by (2.1) by applying our method explained in the Introduction. In the following, we briefly review our main results.

It is convenient to transform $\tilde{b}$ into ρ by setting

$$\tilde{b} = \frac{2}{\alpha}\log\rho. \tag{2.2}$$

Then field equations are as follows:

$$\partial_-(\partial_- h - 2\rho^{-1}\partial_-\rho \cdot h + 2\gamma\rho^{-1}\partial_+\rho) = 0, \tag{2.3}$$

$$\partial_-{}^2\rho = 0. \tag{2.4}$$

It is easy to derive Polyakov's equation

$$\partial_-{}^3 h = 0 \tag{2.5}$$

from (2.3) with the help of (2.4). It should be noted, however, that (2.3) *cannot* be reproduced from (2.4) and (2.5). Owing to (2.4) and (2.5), we may write

$$\rho(x) = a(x^+) + c(x^+)x^-, \tag{2.6}$$

$$h(x) = j^{(1)}(x^+) - 2j^{(0)}(x^+)x^- + j^{(-1)}(x^+)(x^-)^2. \tag{2.7}$$

Regarding x^- as the time variable, we can consistently carry out canonical quantization. Since the x^- dependence is known as in (2.6) and (2.7), it is straightforward to calculate the two-dimensional commutators from the equal-time ones. We find

$$[\rho(x),\ \rho(y)] = 0, \tag{2.8}$$

$$[h(x),\ \rho(y)] = -\frac{1}{2}i\alpha(x^- - y^-)\rho(x)\delta(x^+ - y^+), \tag{2.9}$$

$$[h(x),\ h(y)] = -i\alpha(x^- - y^-)h(x,y)\delta(x^+ - y^+) + \frac{1}{2}i\gamma\alpha(x^- - y^-)^2\delta'(x^+ - y^+), \tag{2.10}$$

where

$$\begin{aligned} h(x,y) &= h(x^-,\, y^-;\, x^+ = y^+) \\ &= j^{(1)} - j^{(0)}(x^- + y^-) + j^{(-1)}x^-y^- \\ &= \frac{1}{2}\left[h(x) + h(y) - \frac{x^- - y^-}{2}\Big(\partial_- h(x) - \partial_- h(y)\Big)\right]_{x^+ = y^+}. \end{aligned} \tag{2.11}$$

The operator solution (2.8)∼(2.10) are uniquely characterized as the solutions to the Cauchy problems for two-dimensional commutators, which are set up by using (2.3)∼(2.5) together with the canonical commutation relations:

$$\begin{cases} (\partial_-{}^x)^2[\rho(x),\ \rho(y)] = 0, \\ \qquad [\rho(x),\ \rho(y)]|_{x^- = y^-} = 0, \\ \quad \partial_-{}^x[\rho(x),\ \rho(y)]|_{x^- = y^-} = 0; \end{cases} \tag{2.12}$$

$$\begin{cases} (\partial_-{}^x)^2[\rho(x),\ h(y)] = 0, \\ \qquad [\rho(x),\ h(y)]|_{x^- = y^-} = 0, \\ \quad \partial_-{}^x[\rho(x),\ h(y)]|_{x^- = y^-} = -i\dfrac{\alpha}{2}\rho\,\delta(x^+ - y^+); \end{cases} \tag{2.13}$$

$$
\left\{
\begin{aligned}
(\partial_-{}^x)^3[h(x),\ h(y)] &= 0,\\
[h(x),\ h(y)]|_{x^-=y^-} &= 0,\\
\partial_-{}^x[h(x),\ h(y)]|_{x^-=y^-} &= -i\alpha h\,\delta(x^+-y^+),\\
(\partial_-{}^x)^2[h(x),\ h(y)]|_{x^-=y^-} &= -i\alpha\partial_- h\cdot\delta(x^+-y^+)+i\gamma\delta'(x^+-y^+).
\end{aligned}
\right.
\tag{2.14}
$$

From (2.12)∼(2.14), we can reproduce (2.8)∼(2.10) by using neither (2.6) nor (2.7). Indeed, (2.8) is the trivial solution to (2.12), while (2.9) and (2.10) are shown to be solutions to (2.13) and (2.14), respectively, by using the following identities:

$$
\begin{aligned}
F(x,y) &= \frac{1}{4\pi}\int d^2u\,\epsilon(x,y;u)D(x-u)(\partial_-{}^u)^2F(u,y)\\
&\quad+\frac{1}{2\pi}\int du^+\Big[D(x-y)\partial_-{}^uF(u,y)\\
&\qquad\qquad -\partial_-{}^uD(x-u)\cdot F(u,y)\Big]_{u^-=y^-},
\end{aligned}
\tag{2.15}
$$

$$
\begin{aligned}
G(x,y) &= \frac{1}{8\pi}\int d^2u\,\epsilon(x,y;u)\tilde{D}(x-u)(\partial_-{}^u)^3G(u,y)\\
&\quad+\frac{1}{4\pi}\int du^+\Big[\tilde{D}(x-y)(\partial_-{}^u)^2G(u,y)\\
&\qquad\qquad -\partial_-{}^u\tilde{D}(x-u)\cdot\partial_-{}^uG(u,y)\\
&\qquad\qquad +(\partial_-{}^u)^2\tilde{D}(x-u)\cdot G(u,y)\Big]_{u^-=y^-},
\end{aligned}
\tag{2.16}
$$

where

$$
\epsilon(x,y;u)\equiv\theta(x^--u^-)-\theta(y^--u^-),
\tag{2.17}
$$

$$
D(\xi)\equiv 2\pi\xi^-\delta(\xi^+),\qquad \tilde{D}(\xi)\equiv\xi^-D(\xi).
\tag{2.18}
$$

From (2.8)∼(2.10), we calculate all multiple commutators explicitly.

Assuming that 1-point functions $\langle\,\rho\,\rangle$ and $\langle\,h\,\rangle$ are constants, we construct all Wightman functions so as to be consistent with all multiple commutators.

Replacing $D^{(+)}$ functions by D_{F} functions, we obtain all n-point τ-functions (i.e., vacuum expectation values of time-ordered products). Here the massless Feynman propagator D_{F} is defined by[4]

$$
D_{\mathrm{F}}(\xi)\equiv\frac{\xi^-}{\xi^+-i0\xi^-}=\xi^-\left(\frac{\theta(\xi^-)}{\xi^+-i0}+\frac{\theta(-\xi^-)}{\xi^++i0}\right),
\tag{2.19}
$$

[4] This definition is unconventional by a factor $-(2\pi)^{-1}$.

which satisfies

$$(2\pi i)\partial_-{}^2 D_{\rm F}(\xi) = \delta^2(\xi). \tag{2.20}$$

Polyakov's WT identities are derived by using (2.5) together with the equal-time commutation relations and (2.20). It is explicitly confirmed that our solution satisfies them. Indeed, our solution is completely consistent with (2.4) and (2.5). It is *not*, however, consistent with (2.3). We thus encounter anomaly. This anomaly cannot be eliminated by renormalizing the values of α and γ.

3. A simple model

The essential feature of the lightcone-gauge two-dimensional quantum gravity is irrelevant to the presence of the last term of (2.1). Hence we may simplify the model by setting $\gamma = 0$. Then (2.1) depends on $\tilde{b}$ only through $\partial_-\tilde{b}$. Hence we can further simplify the model by replacing $\partial_-\tilde{b}$ by b, though this change is quite nontrivial.

Thus we consider a simplified model given by the Lagrangian density

$$\mathcal{L} = b\partial_- h - \frac{\alpha}{2}b^2 h. \tag{3.1}$$

Field equations are

$$\partial_- h - \alpha b h = 0, \tag{3.2}$$

$$\partial_- b + \frac{\alpha}{2}b^2 = 0. \tag{3.3}$$

Setting $b = (2/\alpha)\rho^{-1}$, we reduce (3.3) to

$$\partial_-\rho = 1, \tag{3.4}$$

whence we have

$$\rho(x) = a(x^+) + x^-. \tag{3.5}$$

On the other hand, (3.2) is rewritten as

$$\rho\partial_- h - 2h = 0. \tag{3.6}$$

Multiplying it by $\partial_-{}^2$ and using (3.4), we obtain

$$\partial_-{}^3 h = 0. \tag{3.7}$$

We can therefore express h as in (2.7). In contrast to the lightcone-gauge model, however, $j^{(1)}$, $j^{(0)}$ and $j^{(-1)}$ are not quite independent in the present model. Indeed, substituting (3.5) and (2.7) into (3.6), we find

$$j^{(0)} = -aj, \tag{3.8}$$

$$j^{(1)} = a^2 j, \tag{3.9}$$

where $j \equiv j^{(-1)}$, so that

$$h = \rho^2 j. \tag{3.10}$$

Canonical quantization[5] can be carried out smoothly. We then obtain

$$[a(x^+),\ a(y^+)] = 0, \tag{3.11}$$

$$[j(x^+),\ a(y^+)] = -i\frac{\alpha}{2}\delta(x^+ - y^+), \tag{3.12}$$

$$[j(x^+),\ j(y^+)] = 0; \tag{3.13}$$

therefore,

$$[\rho(x),\ \rho(y)] = 0, \tag{3.14}$$

$$[h(x),\ \rho(y)] = -i\frac{\alpha}{2}\rho^2(x)\delta(x^+ - y^+), \tag{3.15}$$

$$[h(x),\ h(y)] = -i\alpha[\rho(x) - \rho(y)]\rho(x)\rho(y)j \cdot \delta(x^+ - y^+). \tag{3.16}$$

Owing to (3.5) and (3.10), (3.16) is rewritten as

$$[h(x),\ h(y)] = -i\alpha(x^- - y^-)h(x,y)\delta(x^+ - y^+). \tag{3.17}$$

Thus $[h(x),\ h(y)]$ is formally the same as the $\gamma = 0$ case of (2.10).

Here, one should note that (3.14), (3.15) and (3.17) are directly obtained also as the solutions to the Cauchy problems for two-dimensional commutators by using neither (3.5) nor (3.10) in the same way as done in Section 2. In this sense, the natural expression for $[h(x),\ h(y)]$ is (3.17), but not (3.16), although they are equivalent at the level of operator solution.

[5] b is not regarded as a canonical variable.

Nonvanishing multiple commutators are only those which involve at most one ρ. Furthermore, as is seen from (3.17), the multiple commutators involving h's only have the same expression as those in the $\gamma = 0$ case of the lightcone-gauge two-dimensional quantum gravity. The multiple commutators involving one ρ can be easily calculated by using (3.15) and the above result.

1-point functions are arbitrary. For simplicity, we set

$$\langle\, a\,\rangle = 0, \qquad \langle\, j\,\rangle = \text{const.} \tag{3.18}$$

Then we have

$$\langle\, \rho(x)\,\rangle = x^-, \qquad \langle\, h(x)\,\rangle = \langle\, j\,\rangle (x^-)^2. \tag{3.19}$$

Thus *translational invariance is spontaneously broken*. This means that the conventional perturbative approach is *not applicable* to our model, because perturbation theory automatically respects translational invariance as long as Feynman propagators are translationally invariant.

We present some of truncated Wightman functions (a subscript T indicates truncation):

$$\langle\, \rho(x_1)h(x_2)\,\rangle_{\mathrm{T}} = \frac{\alpha}{4\pi}(x_2^-)^2\partial_-{}^{x_1}D^{(+)}(x_1 - x_2), \tag{3.20}$$

$$\langle\, h(x_1)h(x_2)\,\rangle_{\mathrm{T}} = -\frac{\alpha}{2\pi}\langle\, j\,\rangle x_1^- x_2^- D^{(+)}(x_1 - x_2); \tag{3.21}$$

$$\begin{aligned}\langle\, \rho(x_1)h(x_2)h(x_3)\,\rangle_{\mathrm{T}} = \frac{\alpha^2}{8\pi^2}\Big[&x_2^-(x_3^-)^2\partial_- D^{(+)}(x_1 - x_2)\partial_- D^{(+)}(x_2 - x_3) \\ &-(x_2^-)^2 x_3^-\partial_- D^{(+)}(x_1 - x_3)\partial_- D^{(+)}(x_2 - x_3)\Big],\end{aligned} \tag{3.22}$$

$$\begin{aligned}&\langle\, h(x_1)h(x_2)\cdots h(x_n)\,\rangle_{\mathrm{T}} \\ &\quad = \left(-\frac{\alpha}{4\pi}\right)^{n-1}\langle\, j\,\rangle \sum_{\mathrm{P}(i_1\cdots i_n)}^{n!} x_{i_1}^- x_{i_n}^- D_<^{(+)}(x_{i_1} - x_{i_2})\cdots D_<^{(+)}(x_{i_{n-1}} - x_{i_n}),\end{aligned} \tag{3.23}$$

where

$$D^{(+)}(\xi) = \frac{\xi^-}{\xi_+ - i0} \tag{3.24}$$

and $D_<^{(+)}(x_i - x_j)$ equals $D^{(+)}(x_i - x_j)$ for $i < j$ and $D^{(+)}(x_j - x_i)$ for $i > j$; $\mathrm{P}(i_1 \cdots i_n)$ denotes a permutation of $(1, \ldots, n)$.

Now, we proceed to the anomaly problem. It is easy to confirm that all Wightman functions are consistent with (3.4) and (3.7). But the same is not true for (3.6).

We calculate the *nontruncated*[6] function

$$\begin{aligned}\langle\, \rho(x)\partial_- h(z)\cdot h(y)\,\rangle \;&=\; \langle\, \rho(x)\partial_- h(z)\cdot h(y)\,\rangle_{\mathrm{T}} + \langle\, \rho(x)\,\rangle \partial_-{}^{z}\langle\, h(z)h(y)\,\rangle_{\mathrm{T}} \\ &\quad +\partial_-\langle\, h(z)\,\rangle\cdot\langle\, \rho(x)h(y)\,\rangle_{\mathrm{T}} + \partial_-{}^{z}\langle\, \rho(x)h(z)\,\rangle_{\mathrm{T}}\,\langle\, h(y)\,\rangle \\ &\quad +\langle\, \rho(x)\,\rangle\partial_-\langle\, h(z)\,\rangle\cdot\langle\, h(y)\,\rangle. \end{aligned} \tag{3.25}$$

By using the generalized normal-product rule, which implies to drop all terms involving $D^{(+)}(0)$ or its derivative, we find

$$\begin{aligned}\langle\, \rho(x)\partial_- h(x)\cdot h(y)\,\rangle \;&=\; -\frac{\alpha^2}{4\pi^2}x^- y^- [\partial_- D^{(+)}(x-y)]^2 \\ &\quad -2\frac{\alpha}{2\pi}\langle\, j\,\rangle x^- y^- D^{(+)}(x-y) + 2\langle\, j\,\rangle^2 (x^-)^2 (y^-)^2, \end{aligned} \tag{3.26}$$

where use has been made of an identity

$$\xi^- \partial_- D^{(+)}(\xi) = D^{(+)}(\xi). \tag{3.27}$$

The sum of the last two terms of (3.26) is equal to $2\langle\, h(x)h(y)\,\rangle$. Hence we have

$$\begin{aligned}\langle\, [\rho(x)\partial_- h(x) - 2h(x)]h(y)\,\rangle \;&=\; -\frac{\alpha^2}{4\pi^2}x^- y^- [\partial_- D^{(+)}(x-y)]^2 \\ &\neq\; 0. \end{aligned} \tag{3.28}$$

Thus the field equation (3.6) is violated at the level of representation.

4. An equivalent free-field theory

By partial integration, the Lagrangian density (3.1) is equivalent to

$$\tilde{\mathcal{L}} = -(\partial_- b + \frac{\alpha}{2} b^2)h. \tag{4.1}$$

We make the transformation

$$b = (2/\alpha)\rho^{-1}, \qquad h = \rho^2 j \tag{4.2}$$

at the Lagrangian level. We then obtain

$$\tilde{\mathcal{L}} = (2/\alpha)(\partial_- \rho - 1)j. \tag{4.3}$$

[6] For composite-operator calculation, nontruncated functions are necessary to be considered.

This is a free-field theory! Furthermore, the Jacobian of the transformation (4.2) is $-2/\alpha$; that is, it is *nonsingular*. Hence our model should be equivalent to a free-field theory.

Field equations are

$$\partial_- \rho = 1, \qquad \partial_- j = 0. \tag{4.4}$$

Two-dimensional commutators are the same as $(3 \cdot 11) \sim (3 \cdot 13)$, that is,

$$[j(x),\ \rho(y)] = -i\frac{\alpha}{2}\delta(x^+ - y^+) \tag{4.5}$$

and the others vanish.

For a free-field theory, it is sufficient to give 1-point and 2-point functions only. In conformity with (3.18), 1-point functions are set equal to

$$\langle\, \rho(x)\,\rangle = x^-, \qquad \langle\, j(x)\,\rangle = \langle\, j\,\rangle. \tag{4.6}$$

The only nonvanishing truncated 2-point function is

$$\langle\, \rho(x_1) j(x_2)\,\rangle_{\mathrm{T}} = \frac{\alpha}{4\pi}\partial_- D^{(+)}(x_1 - x_2). \tag{4.7}$$

It is now straightforward to calculate Wightman functions involving $h(x)$, by replacing it as a composite field $\rho(x)^2 j(x)$. We then have

$$\begin{aligned}\langle\, \rho(x)\partial_- h(x)\cdot h(y)\,\rangle &= \langle\, \rho(x)\cdot 2\rho(x)\partial_-\rho(x)\cdot j(x)\rho^2(y)j(y)\,\rangle \\ &\quad + \langle\, \rho(x)\cdot \rho^2(x)\partial_- j(x)\cdot \rho^2(y)j(y)\,\rangle.\end{aligned} \tag{4.8}$$

Evidently, the first term of (4.8) equals $2\langle\, h(x)h(y)\,\rangle$. Since (4.6) and (4.7) are consistent with $\partial_- j = 0$, the second term of (4.8) vanishes. Thus we obtain

$$\langle\, [\rho(x)\partial_- h(x) - 2h(x)]h(y)\,\rangle = 0, \tag{4.9}$$

that is, we encounter *no anomaly*. This result is, of course, inconsistent with (3.28). What is the cause of the discrepancy?

We explicitly calculate $\langle\, h(x)h(y)\,\rangle$ by regarding $h(x)$ as a composite field $\rho^2(x)j(x)$ and by applying the conventional normal-product rule.[7] We then obtain

$$\begin{aligned}\langle\, h(x)h(y)\,\rangle &= -\frac{\alpha^2}{4\pi^2}x^- y^- [\partial_- D^{(+)}(x-y)]^2 \\ &\quad -\frac{\alpha}{2\pi}\langle\, j\,\rangle x^- y^- D^{(+)}(x-y) + \langle\, j\,\rangle^2 (x^-)^2 (y^-)^2.\end{aligned} \tag{4.10}$$

[7] When the generalized normal-product rule is applied to a free-field theory, the conventional normal-product rule is reproduced, as it should be.

Compared with the result obtained in Section 3, the first term of (4.10) is an *extra* contribution. Thus the free-field theory is *different* from the model presented in Section 3 at the level of representation.

From (4.10), we have

$$\begin{aligned}\langle\, [h(x),\ h(y)]\,\rangle &= i\frac{\alpha^2}{2\pi}x^- y^- \delta'(x^+ - y^+)\\ &\quad -i\alpha\langle\, j\,\rangle x^- y^-(x^- - y^-)\delta(x^+ - y^+). \end{aligned} \tag{4.11}$$

Is this formula consistent with (3.16)? Evidently, if $x^+ = y^+$ is executed for the operator product, we obtain the last term of (4.11) only, as was done in Section 3. But if we postpone the execution of $x^+ = y^+$ for a moment, the calculation in terms of ρ and j yields

$$\begin{aligned}\langle\, [h(x),\ h(y)]\,\rangle &= [\langle\, \rho(x)j(x)\rho^2(y)\,\rangle - (\ x\ \leftrightarrow\ y\ )]\cdot i\alpha\delta(x^+ - y^+)\\ &= -i\frac{\alpha^2}{2\pi}x^- y^-\left[\frac{1}{x^+ - y^+ - i0} - \frac{1}{y^+ - x^+ - i0}\right]\delta(x^+ - y^+)\\ &\quad -i\alpha\langle\, j\,\rangle x^- y^-(x^- - y^-)\delta(x^+ - y^+). \end{aligned} \tag{4.12}$$

Since the quantity in the square bracket equals $2\dfrac{\mathrm{P}}{x^+ - y^+}$, where P denotes Cauchy principal value, (4.12) reproduces (4.11) if we use the formula

$$\frac{\mathrm{P}}{\xi}\cdot\delta(\xi) = -\frac{1}{2}\delta'(\xi). \tag{4.13}$$

Thus the presence or absence of the first term of (4.11) is the consequence of the violation of the associative law in the product of distributions.

The above result is reconfirmed by considering the c-number term in the commutator $[\,{:}\rho^2(x)j(x){:},\ {:}\rho^2(y)j(y){:}\,]$, where $:\quad:$ denotes the conventional normal product for free fields. One should note, however, that (4.11) is no longer consistent with the equal-time commutation relations for $[h(x),\ h(y)]|_{x^- = y^-}$ and $\partial_-{}^x[h(x),\ h(y)]|_{x^- = y^-}$ [see (2.14)] in the system of h and ρ described in Section 3. A well-known example of this type of pathology is the calculation of the Schwinger term based on the canonical anticommutation relations.

The above justification of the presence of the first term of (4.11) is, of course, heavily based on the specialty of the free-field theory. It is impossible to extend such a prescription to more general framework.

5. Ward-Takahashi identities

In this section, we discuss the τ-functions. They satisfy the WT identities, which are derived from (3.4) and (3.7) together with equal-time commutators. Therefore, their validity is independent of the anomaly problem. On the other hand, we have found two different sets of n-point functions in Section 3 and in Section 4. Is it possible that both sets satisfy the same WT identities?

It is straightforward to derive the following WT identities:

$$\begin{aligned}&\langle\, \mathrm{T}\, \rho(x_1)h(x_2)\cdots h(x_n)\,\rangle_{\mathrm{C}} \\ &\quad = \frac{\alpha}{4\pi}\sum_{k=2}^{n}\partial_-{}^{x_1}D_{\mathrm{F}}(x_1-x_k)\langle\, \mathrm{T}\, \rho^2(x_k)h(x_2)\cdots \widehat{h(x_k)}\cdots h(x_n)\,\rangle_{\mathrm{C}}\,,\end{aligned} \tag{5.1}$$

with

$$\begin{aligned}&\langle\, \mathrm{T}\, \rho^2(x_1)h(x_2)\cdots h(x_n)\,\rangle_{\mathrm{C}} \\ &\quad = \sum_{\mathrm{div}}\langle\, \mathrm{T}\, \rho(x_1)h(x_{i_1})\cdots\rangle_{\mathrm{C}}\,\langle\, \mathrm{T}\, \rho(x_1)h(x_{\ell_1})\cdots\rangle_{\mathrm{C}}\,,\end{aligned} \tag{5.2}$$

$$\begin{aligned}&\langle\, \mathrm{T}\, h(x_1)h(x_2)\cdots h(x_n)\,\rangle_{\mathrm{C}} \\ &\quad = -\frac{\alpha}{4\pi}\sum_{k=2}^{n}D_{\mathrm{F}}(x_1-x_k)[(x_1^- - x_k^-)\partial_-{}^{x_k}+2]\langle\, \mathrm{T}\, h(x_2)\cdots h(x_n)\,\rangle_{\mathrm{C}} \quad \text{for } n \geqq 2,\end{aligned} \tag{5.3}$$

where T, C and ^ denote time-ordered product, connected part and omission, respectively, and $\{\,(i_1,\,\ldots),\,(\ell_1,\,\ldots)\,\}$ is a partition of $(2,\,\ldots,\,n)$. The derivation of (5.3) is the same as in I.

As for $\langle\, \mathrm{T}\, \rho h\cdots h\,\rangle_{\mathrm{C}}$, there is no problem; both sets of the τ-functions are the same. Hence we concentrate our attention to $\langle\, \mathrm{T}\, hh\cdots h\,\rangle_{\mathrm{C}}$.

From (3.23), we have

$$\begin{aligned}&\langle\, \mathrm{T}\, h(x_1)h(x_2)\cdots h(x_n)\,\rangle_{\mathrm{C}}{}^{(h,\,\rho)} \\ &= \left(-\frac{\alpha}{4\pi}\right)^{n-1}\langle\, j\,\rangle \sum_{\mathrm{P}(i_1\cdots i_n)}^{n!} x_{i_1}^- x_{i_n}^- D_{\mathrm{F}}(x_{i_1}-x_{i_2})\cdots D_{\mathrm{F}}(x_{i_{n-1}}-x_{i_n}),\end{aligned} \tag{5.4}$$

where a superscript $(h,\,\rho)$ indicates that the calculation is made on the basis of (3.1).

We can prove that (5.4) satisfies (5.3). The proof is done essentially in the same way as done in I. The only new situation is the presence of the end-point factors $x_{i_1}^- x_{i_n}^-$. If x_j is an end point, we encounter a factor

$$[(x_1^- - x_j^-)\partial_-{}^{x_j}+1]x_j^- = x_1^- \tag{5.5}$$

[The square-bracket factor of (5.5) corresponds to the second part of $(7\cdot 12)$ of I in the end-point case.] That is, the old end-point factor x_j^- is replaced by the new one x_1^-, as it should be.

Now, we consider the τ-functions resulting from the free-field theory (4.3), to which we attach a superscript $(j,\,\rho)$. We can easily see that

$$\langle\, \mathrm{T}\, h(x_1)\cdots h(x_n)\,\rangle_{\mathrm{C}}{}^{(j,\,\rho)} = \langle\, \mathrm{T}\, h(x_1)\cdots h(x_n)\,\rangle_{\mathrm{C}}{}^{\mathrm{tree}} + \langle\, \mathrm{T}\, h(x_1)\cdots h(x_n)\,\rangle_{\mathrm{C}}{}^{\mathrm{loop}} \quad \text{for } n \geqq 2, \qquad (5.6)$$

with

$$\langle\, \mathrm{T}\, h(x_1)\cdots h(x_n)\,\rangle_{\mathrm{C}}{}^{\mathrm{tree}} = \sum_{k=1}^{n} \langle\, j\,\rangle \langle\, \mathrm{T}\, \rho^2(x_k) h(x_1)\cdots \widehat{h(x_k)}\cdots h(x_n)\,\rangle_{\mathrm{C}}\,, \qquad (5.7)$$

$$\langle\, \mathrm{T}\, h(x_1)\cdots h(x_n)\,\rangle_{\mathrm{C}}{}^{\mathrm{loop}} = -\left(\frac{\alpha}{2\pi}\right)^n \prod_{k=1}^{n} x_k^- \sum_{\mathrm{C}(i_1\cdots i_n)}^{(n-1)!} \partial_- D_{\mathrm{F}}(x_{i_1}-x_{i_2})\cdots \partial_- D_{\mathrm{F}}(x_{i_n}-x_{i_1}), \qquad (5.8)$$

where $C(i_1\,\cdots\,i_n)$ denotes a permutation of cyclic ordering $(1,\,2,\,\ldots,\,n)$.[8] Diagrammatically, (5.7) is a sum over the contributions of the tree graphs in which no vertex has the degree more than 3. Because of (4.7), the propagator is $\partial_- D_{\mathrm{F}}$ but not D_{F} in contrast to (5.4). By explicit calculation, we have confirmed that

$$\langle\, \mathrm{T}\, h(x_1)\cdots h(x_n)\,\rangle_{\mathrm{C}}{}^{\mathrm{tree}} = \langle\, \mathrm{T}\, h(x_1)\cdots h(x_n)\,\rangle_{\mathrm{C}}{}^{(h,\,\rho)} \qquad (5.9)$$

for $n = 1,\,2,\,3$. It is quite likely that (5.9) holds for n general.

The WT identity (5.3) shows that the number of propagators increases by one as n increases by one. Accordingly, if $\langle\, \mathrm{T}\, h\cdots h\,\rangle_{\mathrm{C}}{}^{(j,\,\rho)}$ satisfies (5.3), then both $\langle\, \mathrm{T}\, h\cdots h\,\rangle_{\mathrm{C}}{}^{\mathrm{tree}}$ and $\langle\, \mathrm{T}\, h\cdots h\,\rangle_{\mathrm{C}}{}^{\mathrm{loop}}$ must satisfy it separately. Since the former is all right provided that (5.9) is true, we have only to consider the latter.

For $n=2$, (5.8) becomes

$$\langle\, \mathrm{T}\, h(x_1)h(x_2)\,\rangle_{\mathrm{C}}{}^{\mathrm{loop}} = -\left(\frac{\alpha}{2\pi}\right)^2 x_1^- x_2^- [\partial_- D_{\mathrm{F}}(x_1-x_2)]^2. \qquad (5.10)$$

Since we must set $\langle\, \mathrm{T}\, h(x_1)\,\rangle_{\mathrm{C}}{}^{\mathrm{loop}} = 0$, (5.10) should vanish if it obeys the WT identity (5.3). This is evidently impossible. Higher-point functions also do not satisfy (5.3).

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[8] In I, "cyclic permutation" should be read as "permutation of cyclic ordering".

We thus conclude that *the free-field theory violates the WT identities*. This result is understandable as a consequence of the violation of the equal-time commutation relations for h in the free-field theory.

6. Discussion

In the present paper, we have investigated a simple model analogous to the lightcone-gauge two-dimensional quantum gravity. In spite of its simplicity, the former exhibits the anomaly of the same type as in the latter. It is very interesting, however, that our model can be transformed, by a nonsingular transformation, into a free-field theory, which is free of anomaly. We have analyzed the cause of the discrepancy and found that *it is the violation of the associative law for the product of distributions* as is encountered also in the analysis of the Schwinger term.

We have considered the WT identities of our model, and found that the free-field theory does not satisfy them. Accordingly, we should conclude that *the free-field theory cannot be regarded as the anomaly-free version of the original model.*

Two-dimensional massless field theories are, in general, very peculiar models. It is not clear whether or not the pathological phenomena which we have found are characteristic to two-dimensional massless field theories. It is desirable to make further investigation by means of a variety of models.

Appendix. Symmetry properties

Polyakov [17] found an $SL(2, \boldsymbol{R})$ current algebra based on $j^{(1)}(x^+)$, $j^{(0)}(x^+)$ and $j^{(-1)}(x^+)$ in his "induced" quantum gravity. This result can be understood as the x^+-dependent $SL(2, \boldsymbol{R})$ symmetry whose generators are essentially given by $j^{(1)}$, $j^{(0)}$ and $j^{(-1)}$. In our local version of the model, we have found the existence of an extremely huge x^+-dependent symmetry [18]; Polyakov's $SL(2, \boldsymbol{R})$ is no more than a very very tiny subalgebra of it. Since our model considered in Section 3 is analogous to the lightcone-gauge two-dimensional quantum gravity, we can find the existence of a similar huge x^+-dependent symmetry.

Because of (3.8) and (3.9), $j^{(0)}$ and $j^{(1)}$ are no longer independent of a and $j \equiv j^{(-1)}$. The generator Q of our huge algebra is formed from a and j only:

$$Q = \frac{2}{\alpha} \int dx^+ \, F(x^+, \, \boldsymbol{a}, \, \boldsymbol{j}), \tag{A.1}$$

where F is an arbitrary function, and $\boldsymbol{a}$ denotes a set of $a \equiv \rho - x^-$ and its finite-order x^+-derivatives, $\boldsymbol{j}$ being similar. By the generator Q, a and j are transformed as follows:

$$\delta a \equiv i\varepsilon[Q, \ a] = +\varepsilon \left(\frac{\delta}{\delta \boldsymbol{j}} \right)_+ F, \tag{A.2}$$

$$\delta j \equiv i\varepsilon[Q, \ j] = -\varepsilon \left(\frac{\delta}{\delta \boldsymbol{a}} \right)_+ F, \tag{A.3}$$

where

$$\left(\frac{\delta}{\delta \boldsymbol{a}} \right)_+ F \equiv \sum_\ell (-1)^\ell \partial_+{}^\ell \left(\frac{\partial F}{\partial(\partial_+^\ell a)} \right). \tag{A.4}$$

Under this transformation, the action $\int d^2x \, \mathcal{L} = \int d^2x \, \tilde{\mathcal{L}}$ is invariant because

$$\begin{aligned} \delta\tilde{\mathcal{L}} \;\; &= \;\; \varepsilon\frac{2}{\alpha}\partial_- \left[-F + j \left(\frac{\delta}{\delta \boldsymbol{j}} \right)_+ F \right] \\ &\quad + \varepsilon\frac{2}{\alpha}\partial_+ \left[\sum_{\varphi=a,\,j} \sum_\ell \sum_{m=1}^{\ell} (-1)^{m+1} \partial_+{}^{\ell-m} \partial_-\varphi \cdot \partial_+{}^{m-1} \frac{\partial F}{\partial(\partial_+^\ell \varphi)} \right]. \end{aligned} \tag{A.5}$$

The Noether current J^μ is given by

$$J^- = \frac{2}{\alpha} F, \qquad J^+ = 0, \tag{A.6}$$

in which use has been made of $\partial_- a = \partial_- j = 0$.

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